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The p-hyponormality of the Aluthge transform

東北大学・理学研究科
吉野 崇 (Takashi Yoshino)

For a bounded linear operator T on a Hilbert space $\mathcal{H}$ and for some p such as $p > 0$, if $(T^*T)^p \geq (TT^*)^p$, then T is called to be p-hyponormal. For an operator T with its polar decomposition $T = U|T| = |T^*|U$, we say that $T(s,t) = |T|^s U |T|^t$ defined for any s and t such as $s \geq 0$ and $t \geq 0$ is the Aluthge transform of T. About the q-hyponormality of the Aluthge transform $T(s,t)$ of a p-hyponormal operator T, we have the following. In the case where $s = t = \frac{1}{2}$, this result is proved by A. Aluthge ([1]) under the condition that U is unitary.

Theorem. Let T be p-hyponormal for some p such as $p > 0$. Then, for any s and t such as $\max(s,t) \leq p$, $T(s,t)$ is 1-hyponormal and, for any s and t such as $\max(s,t) > p$, $T(s,t)$ is $\frac{p+\min(s,t)}{s+t}$-hyponormal.

To prove our theorem we need the following lemmas which are the slight modifications of known results ([2], [3]).

Lemma 1. If $A \geq B \geq O$, then

$$(1) \quad (B^\gamma A^\alpha B^\gamma)^{\beta\delta} \geq (B^\gamma B^\alpha B^\gamma)^{\beta\delta}$$

and

$$(2) \quad (A^\gamma A^\alpha A^\gamma)^{\beta\delta} \geq (A^\gamma B^\alpha A^\gamma)^{\beta\delta}$$

for each α, β, γ and δ such as

$$\frac{1}{\beta} \leq \alpha, \quad 0 < \beta < 1, \quad \gamma = \frac{\alpha\beta - 1}{2(1-\beta)} \quad \text{and} \quad 0 \leq \delta \leq 1.$$

(In the case where $\gamma = 0$, it is well-known as Heinz's inequality).

Lemma 2. If $A \geq O$ and $\|B\| \leq 1$, then

$$(B^*AB)^\delta \geq B^*A^\delta B \quad \text{for each } \delta \text{ such as} \quad 0 \leq \delta \leq 1.$$

Proof of Theorem Since T is p-hyponormal, $|T|^{2p} \geq |T^*|^{2p}$. Let $\max(s,t) \leq p$. Then we have

$$\begin{aligned} T(s,t)^*T(s,t) &= |T|^tU^*|T|^{2s}U|T|^t = |T|^tU^*(|T|^{2p})^{\frac{s}{p}}U|T|^t \\ &\geq |T|^tU^*(|T^*|^{2p})^{\frac{s}{p}}U|T|^t \quad \text{by Heinz's inequality} \\ &= |T|^tU^*|T^*|^{2s}U|T|^t = |T|^t|T|^{2s}|T|^t \\ &= |T|^s(|T|^{2p})^{\frac{t}{p}}|T|^s \\ &\geq |T|^s(|T^*|^{2p})^{\frac{t}{p}}|T|^s \quad \text{by Heinz's inequality} \\ &= |T|^s|T^*|^{2t}|T|^s = |T|^sU|T|^{2t}U^*|T|^s = T(s,t)T(s,t)^*. \end{aligned}$$

Since, in the case where $p < s$,

$$\begin{aligned} \{U^*|T|^{2s}U\}^{\frac{p}{s}} &\geq U^*\{|T|^{2s}\}^{\frac{p}{s}}U \quad \text{by Lemma 2} \\ &= U^*|T|^{2p}U \\ &\geq U^*|T^*|^{2p}U = |T|^{2p}, \end{aligned}$$

we have

$$\begin{aligned} \{T(s,t)^*T(s,t)\}^{\frac{p+t}{s+t}} &= \{|T|^tU^*|T|^{2s}U|T|^t\}^{\frac{p+t}{s+t}} \\ &= \left[(|T|^{2p})^{\frac{t}{2p}}\left\{(U^*|T|^{2s}U)^{\frac{p}{s}}\right\}^{\frac{s}{p}}(|T|^{2p})^{\frac{t}{2p}}\right]^{\frac{p+t}{s+t}} \\ &\geq \left[(|T|^{2p})^{\frac{t}{2p}}\left\{|T|^{2p}\right\}^{\frac{s}{p}}(|T|^{2p})^{\frac{t}{2p}}\right]^{\frac{p+t}{s+t}} \\ &= |T|^{2(p+t)} \qquad \text{(i)} \end{aligned}$$

by putting $A,\ B,\ \alpha,\ \beta$ and δ in Lemma 1 (1) as follows

$$A = \{U^*|T|^{2s}U\}^{\frac{p}{s}}, \quad B = |T|^{2p},$$
$$\alpha = \frac{s}{p}, \quad \beta = \frac{p+t}{s+t} \quad \text{and} \quad \delta = 1.$$

Next, let $\max(s,t) > p$. Then, in the case where $\max(s,t) = s$, we have

$$\begin{aligned} \{T(s,t)^*T(s,t)\}^{\frac{p+\min(s,t)}{s+t}} &= \{T(s,t)^*T(s,t)\}^{\frac{p+t}{s+t}} \\ &\geq |T|^{2(p+t)} \quad \text{by (i).} \end{aligned} \tag{ii}$$

If $p \geq t$, then

$$\begin{aligned} |T|^{2(p+t)} &= \{|T|^s(|T|^{2p})^{\frac{t}{p}}|T|^s\}^{\frac{p+t}{s+t}} \\ &\geq \{|T|^s(|T^*|^{2p})^{\frac{t}{p}}|T|^s\}^{\frac{p+t}{s+t}} \quad \text{by Heinz's inequality} \\ &= \{|T|^s|T^*|^{2t}|T|^s\}^{\frac{p+t}{s+t}} = \{|T|^sU|T|^{2t}U^*|T|^s\}^{\frac{p+t}{s+t}} \\ &= \{T(s,t)T(s,t)^*\}^{\frac{p+t}{s+t}} = \{T(s,t)T(s,t)^*\}^{\frac{p+\min(s,t)}{s+t}} \end{aligned}$$

and if $p < t$, then

$$\begin{aligned} |T|^{2(p+t)} &= \left[\left\{(|T|^{2p})^{\frac{s}{2p}}(|T|^{2p})^{\frac{t}{p}}(|T|^{2p})^{\frac{s}{2p}}\right\}^{\frac{p+s}{t+s}}\right]^{\frac{p+t}{p+s}} \\ &\geq \left[\left\{(|T|^{2p})^{\frac{s}{2p}}(|T^*|^{2p})^{\frac{t}{p}}(|T|^{2p})^{\frac{s}{2p}}\right\}^{\frac{p+s}{t+s}}\right]^{\frac{p+t}{p+s}} \\ &= \left\{|T|^s|T^*|^{2t}|T|^s\right\}^{\frac{p+t}{t+s}} = \left\{|T|^sU|T|^{2t}U^*|T|^s\right\}^{\frac{p+t}{t+s}} \\ &= \{T(s,t)T(s,t)^*\}^{\frac{p+t}{t+s}} = \{T(s,t)T(s,t)^*\}^{\frac{p+\min(s,t)}{s+t}} \end{aligned}$$

by putting $A,\ B,\ \alpha,\ \beta$ and δ in Lemma 1 (2) as follows

$$A = |T|^{2p}, \quad B = |T^*|^{2p},$$
$$\alpha = \frac{t}{p}, \quad \beta = \frac{p+s}{t+s} \quad \text{and} \quad \delta = \frac{p+t}{p+s}.$$

And hence, for any $t \geq 0$, we have

$$|T|^{2(p+t)} \geq \{T(s,t)T(s,t)^*\}^{\frac{p+\min(s,t)}{s+t}}. \tag{iii}$$

Therefore, in the case where $\max(s,t) = s$, we have, by (ii) and (iii),

$$\{T(s,t)^*T(s,t)\}^{\frac{p+\min(s,t)}{s+t}} \geq \{T(s,t)T(s,t)^*\}^{\frac{p+\min(s,t)}{s+t}}.$$

In the case where $\max(s,t) = t$, we have

$$\begin{aligned}
\{T(s,t)T(s,t)^*\}^{\frac{p+\min(s,t)}{s+t}} &= \{T(s,t)T(s,t)^*\}^{\frac{p+s}{t+s}} \\
&= \{|T|^s U|T|^{2t}U^*|T|^s\}^{\frac{p+s}{t+s}} = \{|T|^s|T^*|^{2t}|T|^s\}^{\frac{p+s}{t+s}} \\
&= \left\{(|T|^{2p})^{\frac{s}{2p}}(|T^*|^{2p})^{\frac{t}{p}}(|T|^{2p})^{\frac{s}{2p}}\right\}^{\frac{p+s}{t+s}} \\
&\leq \left\{(|T|^{2p})^{\frac{s}{2p}}(|T|^{2p})^{\frac{t}{p}}(|T|^{2p})^{\frac{s}{2p}}\right\}^{\frac{p+s}{t+s}} \\
&= \{|T|^s|T|^{2t}|T|^s\}^{\frac{p+s}{t+s}} = |T|^{2(p+s)} \qquad \text{(iv)}
\end{aligned}$$

by putting $A,\ B,\ \alpha,\ \beta$ and δ in Lemma 1 (2) as follows

$$A = |T|^{2p}, \quad B = |T^*|^{2p},$$
$$\alpha = \frac{t}{p}, \quad \beta = \frac{p+s}{t+s} \quad \text{and} \quad \delta = 1.$$

If $p \geq s$, then

$$\begin{aligned}
|T|^{2(p+s)} &= \{|T|^t|T|^{2s}|T|^t\}^{\frac{p+s}{t+s}} = \{|T|^tU^*|T^*|^{2s}U|T|^t\}^{\frac{p+s}{t+s}} \\
&= \{|T|^tU^*(|T^*|^{2p})^{\frac{s}{p}}U|T|^t\}^{\frac{p+s}{t+s}} \\
&\leq \{|T|^tU^*(|T|^{2p})^{\frac{s}{p}}U|T|^t\}^{\frac{p+s}{t+s}} \quad \text{by Heinz's inequality} \\
&= \{|T|^tU^*|T|^{2s}U|T|^t\}^{\frac{p+s}{t+s}} \\
&= \{T(s,t)^*T(s,t)\}^{\frac{p+s}{t+s}} = \{T(s,t)^*T(s,t)\}^{\frac{p+\min(s,t)}{s+t}}
\end{aligned}$$

and if $p < s$, then

$$\begin{aligned}|T|^{2(p+s)} &= \{|T|^{2(p+t)}\}^{\frac{p+s}{p+t}} \\ &\leq \left[\{T(s,t)^*T(s,t)\}^{\frac{p+t}{s+t}}\right]^{\frac{p+s}{p+t}} \quad \text{by (i) and by Heinz's inequality} \\ &= \{T(s,t)^*T(s,t)\}^{\frac{p+s}{s+t}} = \{T(s,t)^*T(s,t)\}^{\frac{p+\min(s,t)}{s+t}}.\end{aligned}$$

And hence, for any $s \geq 0$, we have

$$|T|^{2(p+s)} \leq \{T(s,t)^*T(s,t)\}^{\frac{p+\min(s,t)}{s+t}}. \tag{v}$$

Therefore, in the case where $\max(s,t) = t$, we have also, by (iv) and (v),

$$\{T(s,t)T(s,t)^*\}^{\frac{p+\min(s,t)}{s+t}} \leq \{T(s,t)^*T(s,t)\}^{\frac{p+\min(s,t)}{s+t}}.$$

References

[1] A. Aluthg, On p-hyponormal operators for $0 < p < 1$, Integr. Equat. Oper. Th., 13(1990) 307–315.

[2] F. Hansen, An inequality, Math. Ann., 246(1980) 249–250.

[3] T. Furuta, $A \geq B \geq O$ assures $(B^rA^pB^r)^{\frac{1}{q}} \geq B^{\frac{p+2r}{q}}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)q \geq p+2r$, Proc. Amer. Math. Soc., 101(1987) 85–88.